

### Section 11.3 The integral test and estimates of Sums.

In general, it is difficult to find the exact value of a convergent Series; in 11.2 we saw two examples of series with exact values, namely, geometric series ( $\sum a \cdot r^n$ ) and telescoping series (such as  $\sum \frac{1}{n(n+1)}$ ). It was easy to sum these series since we had explicit formulas for their partial sums; most often, this is not the case. We may still however ask whether a series is convergent or divergent.

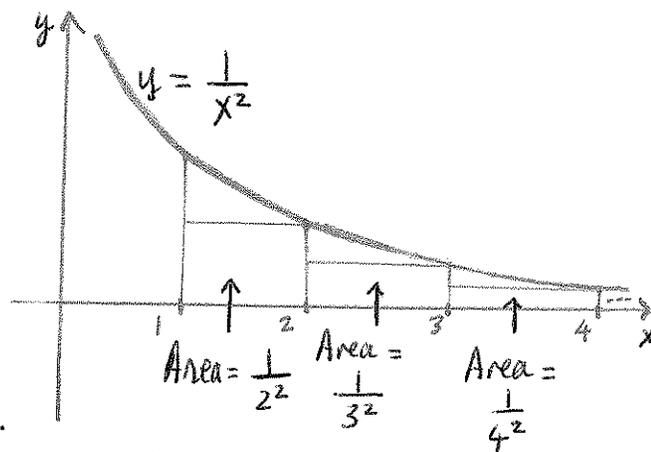
Let's begin by looking at  $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$ . There's no explicit formula for the  $n^{\text{th}}$  partial sum  $S_n$ ; however, using a computer to calculate the  $S_n$ 's, it seems that the partial sums approach a finite value. So, we speculate that the series converges. Here's how we confirm this speculation. Look at the graph of  $y = \frac{1}{x^2}$ . The sum of the areas of

all the rectangles below the graph of  $y = \frac{1}{x^2}$  is  $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \sum_{n=2}^{\infty} \frac{1}{n^2}$ . The sum of these areas is clearly less than the area between  $y = \frac{1}{x^2}$  and the  $x$ -axis, for  $x \geq 1$ .

That is  $\sum_{n=2}^{\infty} \frac{1}{n^2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \leq \int_1^{\infty} \frac{1}{x^2} dx = 1$ .

Therefore,  $\sum_{n=1}^{\infty} \frac{1}{n^2} \leq 1 + \int_1^{\infty} \frac{1}{x^2} dx = 1 + 1 = 2$ , and the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges. This is

because each partial sum is less than 2, and the sequence of partial sums  $\{S_n\}_{n=1}^{\infty}$  is increasing; so  $\{S_n\}_{n=1}^{\infty}$  is monotonically increasing and bounded above by 2, and is therefore convergent (Monotonic sequence theorem).

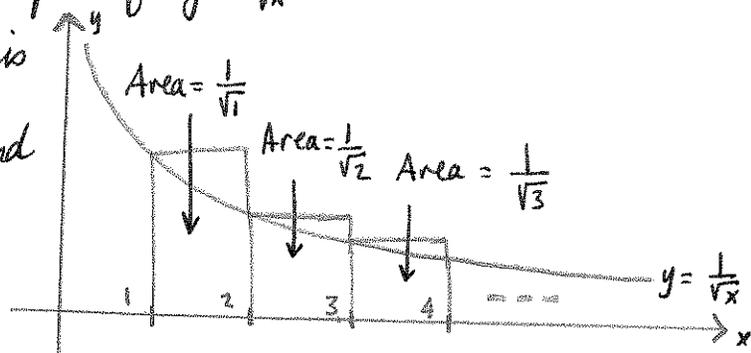


In fact, Euler showed in 1735 that  $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ ! He solved this problem at the age of 28, and it brought him immediate fame, since it withstood the attacks of leading mathematicians of the day.

Let's now look at a different series:  $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots$

Using a computer to calculate its partial sums, we can see that these sums keep increasing without approaching a finite value. So, we speculate that  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$  diverges. To confirm, we look at the graph of  $y = \frac{1}{\sqrt{x}}$ . It is clear that the sum of the areas of the rectangles is greater than the area between  $y = \frac{1}{\sqrt{x}}$  and the x-axis for  $x \geq 1$ . In other words,

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_1^{\infty} \frac{1}{\sqrt{x}} dx = \infty$$



( $\int_1^{\infty} \frac{1}{\sqrt{x}} dx$  diverges by p-test,  $p = \frac{1}{2} < 1$ ). Therefore  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$  and the series diverges.

We now generalize this geometric Argument: **THE INTEGRAL TEST.**

Suppose  $f$  is a continuous, positive, decreasing function on  $[1, \infty)$ . Let  $a_n = f(n)$ .

Then the series  $\sum_{n=1}^{\infty} a_n$  is convergent if and only if  $\int_1^{\infty} f(x) dx$  is convergent. That is,

(i) If  $\int_1^{\infty} f(x) dx$  is convergent, then  $\sum_{n=1}^{\infty} a_n$  is convergent

(ii) If  $\int_1^{\infty} f(x) dx$  is Divergent, then  $\sum_{n=1}^{\infty} a_n$  is divergent.

Remarks: the series does not have to start at  $n=1$ . For example, for the series

$\sum_{n=5}^{\infty} \frac{1}{n^2 \ln(n)}$ , we look at the integral  $\int_5^{\infty} \frac{1}{x^2 \ln x} dx$ . Also, it is necessary that

$f$  is "always decreasing"; as long as  $f$  is "eventually decreasing", we can use the

integral test. Finally, if  $f$  is always negative, we can look at  $(-f)$  instead.

Examples ① Test the series  $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{7n}}$  for convergence or divergence.

The function  $f(x) = \frac{3}{x\sqrt{7x}}$  is continuous (for  $x > 0$ ), positive (for  $x > 0$ ) and decreasing (denominator increases with  $x$ ) for all  $x \geq 1$ . Thus we may use the integral test. Observe that  $\int_1^{\infty} \frac{3}{x\sqrt{7x}} dx = \int_1^{\infty} 3 \cdot 7^{-1/2} \cdot x^{-3/2} dx = \frac{6}{\sqrt{7}}$  (show work!). Thus  $\int_1^{\infty} f(x) dx$  converges, and therefore  $\sum_{n=1}^{\infty} \frac{3}{n\sqrt{7n}}$  converges as well.

Theorem: the "p-Series"  $\sum_{n=1}^{\infty} \frac{1}{n^p}$  converges if  $p > 1$  and diverges if  $p \leq 1$ .

Examples ②  $\sum_{n=1}^{\infty} \frac{1}{n^4}$  converges by p-series,  $p = 4 > 1$

$\sum_{n=1}^{\infty} \frac{1}{5\sqrt{n}}$  diverges by p-series,  $p = \frac{1}{5} < 1$ .

$\sum_{n=1}^{\infty} \frac{1}{n}$  (harmonic series) diverges by p-series,  $p = 1$ .

③ Test the series  $\sum_{n=8}^{\infty} \frac{\ln(n)}{\sqrt{5n}}$  for convergence or divergence. Let  $f(x) = \frac{\ln(x)}{\sqrt{5x}}$

for  $x \geq 8$ ; then  $f(x)$  is positive and continuous ( $x > 0$ ); also, we have

$f'(x) = \frac{2 - \ln(x)}{2x\sqrt{5x}} < 0$  for  $x \geq 8$ , whence  $f(x)$  is decreasing on  $[8, \infty)$ . We

may therefore use the integral test:  $\int f(x) dx = \frac{2}{\sqrt{5}} \sqrt{x} (\ln x - \sqrt{x}) + C$

(integration by parts; show work!). Therefore,  $\int_8^{\infty} f(x) dx = \infty \Rightarrow \sum_{n=8}^{\infty} \frac{\ln(n)}{\sqrt{5n}}$  diverges.

④ Test  $\sum_{n=1}^{\infty} \frac{-4}{n^2+1}$  for convergence or divergence. Let  $f(x) = \frac{-4}{x^2+1}$ , for  $x \geq 1$ .

Notice that  $f(x) < 0$  for all  $x \geq 1$ , so we consider  $-f(x)$  instead. We have

$-f(x) = \frac{4}{x^2+1}$ , continuous, positive, and decreasing (denominator increases w/  $x$ )

for all  $x \geq 1$ . We may therefore use the integral test:  $\int_1^{\infty} \frac{4}{x^2+1} dx = 4 \tan^{-1}(x) \Big|_1^{\infty} = \pi$ .

$\int_1^{\infty} \frac{4}{x^2+1} dx$  converges  $\Rightarrow \sum_{n=1}^{\infty} \frac{4}{n^2+1}$  converges  $\Rightarrow \sum_{n=1}^{\infty} \frac{-4}{n^2+1}$  converges (a constant

multiple of a convergent series is also convergent).

Estimating the sum of a series: if a series converges, we might be interested in estimating its sum (in most cases it is difficult to find the exact sum). Recall

that  $\sum_{n=1}^{\infty} a_n$  converges if the sequence of partial sums  $\{S_n\}_{n=1}^{\infty}$ , where  $S_n = a_1 + a_2 + \dots + a_n$ , converges to a finite value  $S$ . So, if  $n$  is large enough,  $S_n$  is a good approximation to  $S$ ; but how good of an approximation?

We need to estimate the remainder (or error)  $R_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \dots$

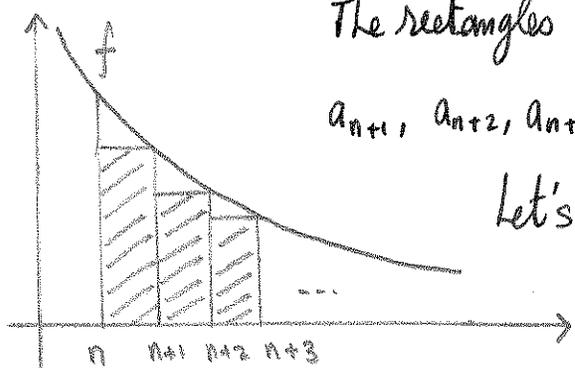
For example, if we use 10 terms to approximate the sum of  $\sum_{n=1}^{\infty} \frac{1}{2^n}$ , we get  $S_{10} = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{10}} = \frac{1023}{1024}$ . We know the exact sum is  $S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$ .

So the remainder (or error) of this approximation is  $R_{10} = 1 - \frac{1023}{1024} = \frac{1}{1024}$ .

In general, we use the same idea as the integral test to estimate  $R_n$ :

Let  $f(x)$  be a positive, continuous, decreasing function on  $[1, \infty)$ . Graph  $f(x)$  for  $x \geq n$ .

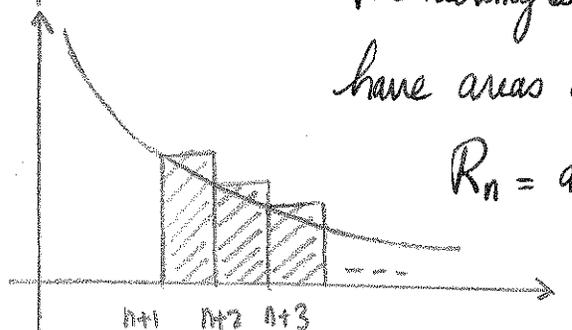
The rectangles below the graph of  $f$ , for  $x \geq n$  have areas  $a_{n+1}, a_{n+2}, a_{n+3}, \dots$ . Therefore  $R_n = a_{n+1} + a_{n+2} + \dots \leq \int_n^{\infty} f(x) dx$ .



Let's do the same trick, this time with rectangles above the graph of  $f$ , starting at  $x = n+1$ .

The rectangles with top edges above the graph of  $f$ , for  $x \geq n+1$  have areas  $a_{n+1}, a_{n+2}, a_{n+3}, \dots$ . Therefore we have

$$R_n = a_{n+1} + a_{n+2} + a_{n+3} + \dots > \int_{n+1}^{\infty} f(x) dx.$$



the two results above: if  $\sum a_n$  converges,  $f(x)$  is

continuous, positive and decreasing,  $f(k) = a_k$ , and  $R_n = S - S_n$ . Then

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx.$$

Examples ① a. Approximate the sum of  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  by using the first 10 terms, and estimate the error of this approximation

b. How many terms are required to ensure the sum is accurate to within 0.0005?

solution: a. we have  $S_{10} = \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{10^2} = 1.549767731$ . Let  $f(x) = \frac{1}{x^2}$ ,  $x \geq 1$ .

Then  $f(x)$  is positive, continuous, and decreasing on  $[1, \infty)$ . By the remainder

theorem,  $R_{10} \leq \int_{10}^{\infty} \frac{1}{x^2} dx = -\frac{1}{x} \Big|_{10}^{\infty} = 0.1$ . So the size of the error is

at most 0.1. (the actual sum is  $\frac{\pi^2}{6} \approx 1.644934068$ , which is within 0.1 of  $S_{10}$ ).

b. we want to find  $n$  such that  $R_n \leq 0.0005$ . we calculate  $n$  such that

$$R_n \leq \int_n^{\infty} \frac{1}{x^2} dx \leq 0.0005. \text{ That is } -\frac{1}{x} \Big|_n^{\infty} \leq 0.0005 \Leftrightarrow \frac{1}{n} \leq 0.0005 \Leftrightarrow n \geq 2000$$

This means we need 2000 terms to get an error smaller than 0.0005. In fact,

$$S_{2000} = 1.644434192, \text{ which confirms our work!}$$

. The remainder estimate is  $\int_{n+1}^{\infty} f(x) dx \leq \frac{R_n}{S - S_n} \leq \int_n^{\infty} f(x) dx$ . Add  $S_n$  to all sides, to get  $S_n + \int_{n+1}^{\infty} f(x) dx \leq S \leq S_n + \int_n^{\infty} f(x) dx$ : Estimate of actual Sum  $S$ .

Examples ② Use this estimate, with  $n=10$ , to estimate  $\sum_{n=1}^{\infty} \frac{1}{n^3}$ .

we have  $S_{10} + \int_{11}^{\infty} \frac{1}{x^3} dx \leq S \leq S_{10} + \int_{10}^{\infty} \frac{1}{x^3} dx$ : we calculate

$$S_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \dots + \frac{1}{10^3} = 1.197531986, \quad \int_{11}^{\infty} \frac{1}{x^3} dx = 0.004132231405, \text{ and}$$

$$\int_{10}^{\infty} \frac{1}{x^3} dx = 0.005; \text{ Therefore}$$

$$1.201664217 \leq S \leq 1.202531986.$$

If we use the midpoint of this interval as an estimate, we get

$S \approx 1.202098102$ , which is within 0.00005 of the actual sum

( $\sum_{n=1}^{\infty} \frac{1}{n^3} = 1.202056903$ ); A very good approximation w/ just 10 terms!